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An assumption-free theorem on discrete-time positive real systems

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Abstract: The discrete-time positive real lemma is examined and the assumptions of controllability and observability are removed. An intermediate theorem is proved that removes only the assumption of controllability. This work is intended to be the counterpart—and is inspired by—recent work on the continuous-time positive real lemma, and does not assume asymptotic stability unlike the recent results of Ferrante and Ntogramatzidis (2017).

Keywords: Linear systems; time invariant systems; linear multivariable systems; discrete-time positive real lemma; positive real systems; passive systems.

1. INTRODUCTION

The positive real lemma is recognised as one of the most fundamental results in systems and control. In continuous-time, there have been many contributions that have sought to relax the controllability and observability assumptions from the classical version of this lemma (e.g., Pandolfi, 2001; Kunimatsu et al., 2008), and related problems on dissipativity and optimal control (e.g., Çamlibel et al. (2003)). This is also so in the closely related bounded real lemma, with the two often being presented hand in hand. Recently, Hughes (2017, 2018) provided versions of the positive real and bounded real lemmas which do not assume controllability or observability nor any alternative superfluous assumptions. Analogous results, however, are yet to appear for the discrete-time positive real and bounded real lemmas and associated problems, yet we cannot say that discrete-time systems are any less important than their continuous time counterparts. Indeed, in discfrete-time, the analogous positive real and bounded real lemmas have found application in areas such as stability analysis, low-sensitivity filter design, solution to the 2-dimensional Lyapunov equation, and signal processing (see Xiao and Hill (1999) and the references therein). The classical presentation of the discrete-time positive real lemma typically makes assumptions of controllability and observability (Hitz and Anderson, 1969). These two issues cannot be trivially neglected in any complete treatment.

Ferrante and Ntogramatzidis (2017) detail some of these positive real results in continuous time and offer the discrete-time counterpart of Pandolfi (2001). Via the discrete-time version (Baggio and Ferrante, 2016, Theorem 2) of the famous spectral factorisation theorem of Youla (1961), they remove the controllability requirement in the proof of the discrete-time positive real lemma. However, they only consider systems with strictly stable eigenvalues.

So there is yet work to be done in finding the discrete-time equivalent of the continuous time results of Hughes (2018). All this serves to motivate the contribution of this paper: the removal of both the controllability and observability assumptions from the discrete-time positive real lemma.

The paper is structured as follows, after listing the required notation, Section 2 details existing and well known results and the important definitions. In Section 3 we lay out the main contributions of this paper: an assumption free stating of the discrete-time positive real lemma. The proofs of the observable and unobservable case follow in Sections 4 and 5 respectively. Finally, Section 6 concludes.

1.1 Notation and definitions

Some notation is in order before proceeding. Firstly, $\mathbb{R}$ ($\mathbb{C}$) denotes the real (complex) numbers; $\mathbb{Z}_+$ are the non-negative integers. $\mathcal{C}_+$ ($\bar{\mathcal{C}}_+$) denotes the space of complex numbers with $|z| > 1$ ($|z| \geq 1$) and $\mathcal{C}_-$ ($\bar{\mathcal{C}}_-$) denotes the complex numbers with $|z| < 1$ ($|z| \leq 1$). If $\lambda \in \mathbb{C}$ then $\bar{\lambda}$ denotes its complex conjugate. We let $\mathbb{R}[z]$ (resp., $\mathbb{R}(z)$, $\mathbb{R}[z, \frac{1}{z}]$) denote the polynomials (resp., rational functions, Laurent polynomials) in the indeterminate z with real coefficients. As for matrices, let $\mathbb{R}^{m\times n}$ (resp., $\mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}[z]$, $\mathbb{R}^{m\times n}(z)$) denote the $m\times n$ matrices with entries from $\mathbb{R}$ (resp., $\mathbb{C}$, $\mathbb{R}[z]$, $\mathbb{R}(z)$), and let $\mathbb{R}_s^{m\times m}$ indicate the space of symmetric matrices. Vectors are denoted in bold font: $\mathbf{v} \in \mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{R}^n[z]$, or $\mathbb{R}^n(z)$. The set of eigenvalues of M is denoted by $\text{spec}(M) := \{\lambda \in \mathbb{C} \mid \det(\lambda I - M) = 0\}$.

If $M \in \mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$, $\mathbb{R}^{m\times n}[z]$, or $\mathbb{R}^{m\times n}(z)$, then M^T denotes its transpose, and if M is nonsingular (i.e., $\det(M) \neq 0$) then M^{-1} denotes its inverse. If $M \in \mathbb{C}^{m\times n}$ then M^* denotes the Hermitian transpose. If $M \in \mathbb{R}^{m\times n}[z]$ or $\mathbb{R}^{m\times n}(z)$ then $M^{\sim}$ satisfies $M^{\sim}(z) = (M(\frac{1}{z}))^T$. The non-negative (positive definite) matrices are denoted by $M \geq 0$ ($M > 0$). I denotes the identity matrix. The notation $\text{col}(M_1, M_2)$ represents the matrix formed by stacking (the appropriately dimensioned) M_1 on top of M_2, while $\text{diag}(M_1, M_2)$ denotes the block diagonal matrix formed by placing M_1 and M_2 along the diagonal. We also define the normal rank of a matrix as $\text{normalrank}(H) := \max_{\lambda\in\mathbb{C}}(\text{rank}(H(\lambda)))$. A matrix is said to be semisimple if

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its Jordan normal form has no blocks of size greater than 1 (i.e., there are no Jordan chains of length greater than 1). A matrix $M(\lambda)$ is said to be unimodular if $\det(M(\lambda))$ is a nonzero constant for all λ.

2. BACKGROUND

We place the paper in context by stating the classical DPR lemma and other relevant existing results. We shall be considering the discrete-time linear state space system

$$\begin{aligned} &\mathbf{x}(k{+}1) = A\mathbf{x}(k) + B\mathbf{u}(k) \text{ and} \\ &\mathbf{y}(k) = C\mathbf{x}(k) + D\mathbf{u}(k),\ \ k = 0, 1, 2, \ldots; \\ &A \in \mathbb{R}^{d\times d}, B \in \mathbb{R}^{d\times n}, C \in \mathbb{R}^{n\times d}, D \in \mathbb{R}^{n\times n}. \end{aligned} \tag{1}$$

We begin with the definition of a discrete-time positive real (DPR) function

Definition 1. (DPR). Let $G \in \mathbb{R}^{n\times n}(z)$. G is DPR if (i) G is analytic in $\mathcal{C}_+$ and (ii) $G(z) + G(z)^* \geq 0$ for all $z \in \mathcal{C}_+$.

We also recall the definitions of controllability, observability and stabilizability. For more system theoretic definitions of these concepts we refer to (Franklin and Powell, 1980, Section 6.7) and (Sarachik and Kreindler, 1965). Of more relevance here are the associated algebraic tests for observability and stabilizability. The system in (1) is observable (and we say the pair (C, A) is observable) if and only if

$$\mathcal{O} = \mathrm{col}(C, CA, \ldots, CA^{d-1}) \tag{2}$$

has full column rank. Also, the system in (1) is stabilizable (and we say the pair (A, B) is stabilizable) if and only if $[\lambda I - A\ B]$ has full row rank for all $\lambda \in \bar{\mathcal{C}}_+$ (Zhou et al., 1996).

We recall the classical discrete-time positive real lemma from Hitz and Anderson (1969).

Lemma 2. Let (A, B, C, D) be as in (1) with (A, B) controllable and (C, A) observable, and let $G(z) = D + C(zI - A)^{-1}B$. Then G is DPR if and only if there exists a $P > 0$, $L \in \mathbb{R}^{r\times d}$ and $W \in \mathbb{R}^{r\times n}$ (for some integer r) such that $P - A^TPA = L^TL$, $C^T - A^TPB = L^TW$, and $D + D^T - B^TPB = W^TW$.

Remark 3. We note that the equivalence of the conditions in Lemma 2 no longer hold if (A, B) is not controllable. For example, let $A = 1, B = 0, C = 1$ and $D = 1$. Then $G(z)D + C(zI - A)^{-1}B = 1$, which is DPR. But $P - A^TPA = 0$, so $P - A^TPA = L^TL$ implies that $L = 0$. We then require that $1 = C^T - A^TPB = L^TW = 0$, a contradiction. Note that this example falls within the class of systems not considered by Ferrante and Ntogramatzidis (2017).

One final relevant result is in order before stating the theorem that is the purpose of this paper. The spectral factorisation theorem of Youla is a longstanding result first shown by Youla (1961), but its discrete-time equivalent is a relatively recent result (Baggio and Ferrante, 2016) that we recall here.

Lemma 4. Let $H \in \mathbb{R}^{n\times n}(z)$ satisfy $H(z) \geq 0$ for all z on the unit circle, with the exception of poles of H, and let $\mathrm{normalrank}(H) = r$. Then there exists $Z \in \mathbb{R}^{r\times n}(z)$ such that (i) $H = Z^\sim Z$; (ii) Z is analytic in $\mathcal{C}_+ \cup \infty$; and (iii) $Z(\lambda)$ has full row rank for all $\lambda \in \mathcal{C}_+ \cup \infty$. If Z satisfies conditions (i)—(iii), then we call Z a discrete-time spectral factor of H, and if H has no poles on the unit circle then Z has no poles on the unit circle.

3. MAIN RESULTS

We state the main contribution of this paper: a theorem on discrete-time positive real systems that requires neither controllability or observability.

Theorem 5. Let $\tilde{A} \in \mathbb{R}^{\tilde{d}\times\tilde{d}}$, $\tilde{B} \in \mathbb{R}^{\tilde{d}\times\tilde{n}}$, $\tilde{C} \in \mathbb{C}^{\tilde{n}\times\tilde{d}}$, and $\tilde{D} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$, and let $\tilde{\mathcal{O}} = \mathrm{col}(\tilde{C}, \tilde{C}\tilde{A}, \ldots, \tilde{C}\tilde{A}^{\tilde{d}-1})$. The following are equivalent:

(1) Let $\tilde{G}(z) = \tilde{D} + \tilde{C}(zI - \tilde{A})^{-1}\tilde{B}$ and let $\tilde{U} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}[z]$ and $\tilde{V} \in \mathbb{R}^{\tilde{n}\times\tilde{d}}[z]$ be left coprime polynomial matrices that satisfy $\tilde{U}(z)\tilde{B}^T\tilde{\mathcal{O}}^T = \tilde{V}(z)(\frac{1}{z}I - \tilde{A}^T)\tilde{\mathcal{O}}^T$.[1] The following hold:
   (a) $\tilde{G}$ is DPR.
   (b) If $\tilde{\mathbf{z}} \in \mathbb{C}^{\tilde{n}\tilde{d}}$ and $\lambda \in \bar{\mathcal{C}}_+$ are such that $\tilde{\mathbf{z}}^T\tilde{\mathcal{O}}[\lambda I{-}\tilde{A}\quad \tilde{B}] = 0$, then $\tilde{\mathbf{z}}^T\tilde{\mathcal{O}} = 0$.
   (c) If $\tilde{\mathbf{b}} \in \mathbb{R}^{\tilde{n}}[z]$ satisfies $\tilde{\mathbf{b}}^T(\tilde{U}(\tilde{V}\tilde{C}^T + \tilde{U}\tilde{D}^T)^\sim + (\tilde{V}\tilde{C}^T + \tilde{U}\tilde{D}^T)\tilde{U}^\sim) = 0$, then there exists $\tilde{\mathbf{w}} \in \mathbb{R}^{\tilde{d}}[z]$ such that $(\tilde{\mathbf{b}}^T\tilde{U})(z)\tilde{C} = \tilde{\mathbf{w}}(z)^T(zI - \tilde{A})$;

(2) There exists $\tilde{P} \in \mathbb{R}_s^{\tilde{d}\times\tilde{d}}$ such that $\tilde{P} \geq 0$ and
$$\begin{bmatrix} \tilde{P} - \tilde{A}^T\tilde{P}\tilde{A} & \tilde{C}^T - \tilde{A}^T\tilde{P}\tilde{B} \\ \tilde{C} - \tilde{B}^T\tilde{P}\tilde{A} & \tilde{D} + \tilde{D}^T - \tilde{B}^T\tilde{P}\tilde{B} \end{bmatrix} \geq 0; \tag{3}$$

(3) There exists $\tilde{P}_- \in \mathbb{R}_s^{\tilde{d}\times\tilde{d}}, \tilde{L} \in \mathbb{R}^{\tilde{r}\times\tilde{d}}$ and $\tilde{W} \in \mathbb{R}^{\tilde{r}\times\tilde{n}}$ such that (i) $\tilde{P}_- \geq 0$; (ii) $\tilde{P}_- - \tilde{A}^T\tilde{P}_-\tilde{A} = \tilde{L}^T\tilde{L}$; (iii) $\tilde{C}^T - \tilde{A}^T\tilde{P}_-\tilde{B} = \tilde{L}^T\tilde{W}$; (iv) $\tilde{D} + \tilde{D}^T - \tilde{B}^T\tilde{P}_-\tilde{B} = \tilde{W}^T\tilde{W}$; and (v) $\tilde{W} + \tilde{L}(zI - \tilde{A})^{-1}\tilde{B}$ is a discrete-time spectral factor of $\tilde{G} + \tilde{G}^\sim$.

Remark 6. Referring back to the example in remark 3, it can be verified that condition 1b of the above theorem is violated for that example. Specifically, in order that $\tilde{\mathbf{z}} \in \mathbb{C}^{\tilde{n}\tilde{d}}$ and $\lambda \in \bar{\mathcal{C}}_+$ satisfy $\tilde{\mathbf{z}}^T\tilde{\mathcal{O}}\left[\lambda I - \tilde{A}\ \tilde{B}\right] = 0$, then $\lambda = 1$, so it is not necessarily the case that $\tilde{\mathbf{z}}^T\tilde{\mathcal{O}} = 0$.

Secondly, let
$$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \tilde{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tilde{C} = [2\ 1] \text{ and } \tilde{D} = 1.$$
This is an example of a system for which A possesses an eigenvalue at 0.5 that is strictly stable and an eigenvalue at 1 that is controllable. In this case, it can be verified that $G(z) = \tilde{D} + \tilde{C}(zI - \tilde{A})^{-1}\tilde{B} = \frac{z+1}{z-1}$, whence $G(z) + G(z)^* = 2\frac{|z|^2-1}{|z-1|^2}$, so G is DPR. Moreover,
$$\tilde{P} - \tilde{A}^T\tilde{P}\tilde{A} = \begin{bmatrix} 0 & 0.5\tilde{P}_{21} \\ 0.5\tilde{P}_{12} & 0.75\tilde{P}_{22} \end{bmatrix},$$
and
$$\tilde{C}^T - \tilde{A}^T\tilde{P}\tilde{B} = \begin{bmatrix} 2 - \tilde{P}_{11} \\ 1 - 0.5\tilde{P}_{21} \end{bmatrix},$$
and we again find that there does not exist a matrix $\tilde{P} \geq 0$ satisfying the properties of conditions 2 or 3 in Theorem 5. In this case, it can be verified that $\tilde{U}(z) = -z + 1$

[1] Note that such matrices $\tilde{U}$ and $\tilde{V}$ will always exist and can be obtained by computing a basis for the left syzygy of $\begin{bmatrix} \tilde{B}^T \\ (\frac{1}{z}I - \tilde{A}^T) \end{bmatrix}\tilde{\mathcal{O}}^T$

and $\tilde{V}(z) = [z\ 0]$ are left coprime polynomial matrices that satisfy $\tilde{U}(z)\tilde{B}^T\tilde{\mathcal{O}}^T = \tilde{V}(z)(\frac{1}{z}I - \tilde{A}^T)\tilde{\mathcal{O}}^T$, whereupon it can be verified that $(\tilde{U}(\tilde{V}\tilde{C}^T + \tilde{U}\tilde{D}^T)^\sim + (\tilde{V}\tilde{C}^T + \tilde{U}\tilde{D}^T)\tilde{U}^\sim) = 0$, and it follows that any polynomial vector $\tilde{\mathbf{b}}$ satisfies $\tilde{\mathbf{b}}^T(\tilde{U}(\tilde{V}\tilde{C}^T + \tilde{U}\tilde{D}^T)^\sim + (\tilde{V}\tilde{C}^T + \tilde{U}\tilde{D}^T)\tilde{U}^\sim) = 0$. It then follows that $(\tilde{\mathbf{b}}^T\tilde{U})(z)\tilde{C} = \tilde{\mathbf{b}}^T\,[2-2z\ \ 1-z]$, which cannot be written in the form $\tilde{\mathbf{w}}(z)^T(zI - \tilde{A})$ for some polynomial vector $\tilde{\mathbf{w}}$. Thus, condition 1c of Theorem 5 is violated for this example.

Our proof of Theorem 5 proceeds by first showing the following theorem that removes the controllability assumption from the DPR lemma, but assumes observability.

Theorem 7. Let A, B, C, and D define a linear state space system as in (1) with the pair (C, A) observable. The following are equivalent:

(1) Let $G(z) = D + C(zI - A)^{-1}B$ and let $U \in \mathbb{R}^{n\times n}[z]$ and $V \in \mathbb{R}^{n\times d}[z]$ be left coprime polynomial matrices that satisfy $U(z)B^T = V(z)(\frac{1}{z}I - A^T)$. The following hold:
  (a) G is DPR.
  (b) (A, B) is stabilizable.
  (c) If $\mathbf{b} \in \mathbb{R}^n[z]$ satisfies $\mathbf{b}^T(U(VC^T + DU^T)^\sim + (VC^T + DU^T)U^\sim) = 0$, then there exists $\mathbf{w} \in \mathbb{R}^d[z]$ such that $(\mathbf{b}^T U)(z)C = \mathbf{w}(z)^T(zI - A)$;

(2) There exists $P \in \mathbb{R}_s^{d\times d}$ such that $P \geq 0$ and
$$\begin{bmatrix} P - A^T PA & C^T - A^T PB \\ C - B^T PA & D + D^T - B^T PB \end{bmatrix} \geq 0; \tag{4}$$

(3) There exists $P_- \in \mathbb{R}_s^{d\times d}, L \in \mathbb{R}^{r\times d}$ and $W \in \mathbb{R}^{r\times n}$ such that (i) $P_- \geq 0$; (ii) $P_- - A^T P_- A = L^T L$; (iii) $C^T - A^T P_- B = L^T W$; (iv) $D + D^T - B^T P_- B = W^T W$; and (v) $W + L(zI - A)^{-1}B$ is a discrete-time spectral factor of $G + G^\sim$.

In particular, if P is as in condition 2 (P_- as in condition 3) then $P > 0$ ($P_- > 0$).

Owing to space constraints, the majority of this paper is concerned with the proof of Theorem 7. The proof of Theorem 5 can then be obtained by relating the system considered in that theorem to the observable subsystem obtained from the staircase observability form. The proof is sketched in Section 5 but will be provided in full in a subsequent paper.

4. PROOF OF THEOREM 7

Before proving Theorem 7 it is necessary to provide an alternative characterization of a DPR function, which leads to a convenient decomposition.

Lemma 8. G is DPR if and only if

(1) If z is not a pole of G and $|z| = 1$, then $G(z) + G^\sim(z) \geq 0$.
(2) If z_0 is a pole of G with $|z_0| = 1$, then it is simple and
$$X_{z_0} := \lim_{z\to z_0}\left(\frac{G(z)(z - z_0)}{z_0}\right)$$
is non-negative definite Hermitian.

Moreover, if G is DPR, and z_k denote the poles of G that satisfy $|z_k| = 1$ and $z_k \neq -1$, then
$$G_1(z) := \frac{X_{-1}}{2}\frac{z-1}{z+1} + \sum_{k=1}^{N} X_{z_k}\frac{z_k(1+z)}{(1 + z_k)(z-z_k)} \tag{5}$$
is DPR and satisfies $G_1 + G_1^\sim = 0$, and $G_2 := G - G_1$ is also DPR.

Proof. That G is DPR if and only if 1 and 2 hold is shown in (Hitz and Anderson, 1969, Lemma 2). Next, note that if $G_1(z)$ is as defined in (5) then it can be verified that $\lim_{z\to z_k}(G_1(z)(z - z_k)/z_k) = X_{z_k}$ and $\lim_{z\to -1}(-G_1(z)(z+1)) = X_{-1}$. We next show that $G_1 \in \mathbb{R}^{n\times n}(z)$ and $G_1 + G_1^\sim = 0$. To see this, we note initially that, since $G \in \mathbb{R}^{n\times n}(z)$, then the poles of G appear in complex conjugate pairs, and if z_k is a pole of G then $X_{\bar{z}_k} = \bar{X}_{z_k}$. It is then easy to show that $G_1 \in \mathbb{R}^{n\times n}(z)$. Also, since X_{-1} is real, then X_{-1} is also symmetric, which implies that
$$\frac{X_{-1}^T}{2}\frac{\frac{1}{z} - 1}{\frac{1}{z} + 1} = -\frac{X_{-1}}{2}\frac{z - 1}{z + 1}.$$
Moreover, since X_{z_k} is Hermitian, then $X_{z_k}^T = \bar{X}_{z_k} = X_{\bar{z}_k}$. Since, in addition, $\bar{z}_k = 1/z_k$, it follows that
$$X_{z_k}^T\frac{z_k(1+\frac{1}{z})}{(1 + z_k)(\frac{1}{z}-z_k)} = -X_{\bar{z}_k}\frac{\bar{z}_k(1 + z)}{(1 + \bar{z}_k)(z - \bar{z}_k)}.$$
From which we can see that the term in G_1 corresponding to $\bar{z}_k$ cancels with the term in $G_1^\sim$ corresponding to z_k in the sum over all poles in $G_1 + G_1^\sim$, including the X_{-1} term. So $G_1 + G_1^\sim = 0$ on the unit circle. Also on the unit circle, $G_1^\sim(z) = G_1(z)^*$, so we have that $G_1(z) + G_1(z)^* = 0$ when z is not a pole, and when z_0 is a pole of G_1 we have $\lim_{z\to z_k}(G_1(z)(z - z_k)/z_k) = X_{z_k} \geq 0$ so G_1 is DPR.

From the limits of G and G_1 above it follows that $\lim_{z\to z_k}((G - G_1)(z)(z - z_k)/z_k) = 0$ and $\lim_{z\to -1}((G - G_1)(z)(z+1)) = 0$. It is also the case that for $|z| = 1$ and z not a pole of G, $G_2(z) + G_2(z)^* = G(z) + G(z)^* \geq 0$ on the unit circle, and by appealing to analytic continuation we see this holds everywhere, and G_2 is therefore DPR by the two conditions at the start of this lemma.

We next prove another intermediate lemma that yields a useful decomposition for the case in which $\mathrm{spec}(A) \notin \mathcal{C}_-$.

Lemma 9. Let A, B, C and D be as in (1) where $G(z) = D + C(zI - A)^{-1}B$ is DPR, (A, B) is stabilizable, and (C, A) is observable. Then there exists a nonsingular $T \in \mathbb{R}^{d\times d}$ such that
$$TAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, TB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, CT^{-1} = [C_1\ C_2], \tag{6}$$
where A_1 is semisimple and $\mathrm{spec}(A_1)$ is on the unit circle, $\mathrm{spec}(A_2) \in \mathcal{C}_-$, and both (C_1, A_1) and (C_2, A_2) are observable. Also, if A_1, B_1, C_1 and A_2, B_2, C_2 satisfy the aforementioned conditions, then

(1) Let X_{-1}, z_k and X_{z_k} be as in Lemma 8, and let $D_2 := D - \frac{1}{2}X_{-1} - \sum_{k=1}^{N}\frac{z_k}{1+z_k}X_{z_k}$. If there exists $P_2 > 0$ such that
$$\begin{bmatrix} P_2 - A_2^T P_2 A_2 & C_2^T - A_2^T P_2 B_2 \\ C_2 - B_2^T P_2 A_2 & D_2 + D_2^T - B_2^T P_2 B_2 \end{bmatrix} \geq 0 \tag{7}$$
then there exists $P > 0$ such that (4) holds.
(2) If there exists $P > 0$ such that (4) holds, then $\hat{P} := (T^T)^{-1}PT^{-1}$ takes the form $\hat{P} = \mathrm{diag}\,(P_1\ P_2)$ where (i) P_1 is uniquely determined by the equations

$P_1 - A_1^T P_1 A_1 = 0$ and $C_1^T - A_1^T P_1 B_1 = 0$; and (ii) with the notation $D_2 := D - \frac{1}{2}B_1^T P_1 B_1$, then $P_2 > 0$ satisfies (7).

Proof. That there is a nonsingular $T \in \mathbb{R}^{d\times d}$ such that TAT^{-1} takes the form of (6) with $\mathrm{spec}(A_1) \in \bar{\mathcal{C}}_+$ and $\mathrm{spec}(A_2) \in \mathcal{C}_-$ is clear from the real Jordan form of A (see (Gantmacher, 1980, Chapter VII)). Let B_1, B_2, C_1 and C_2 be defined as in (6). Since (A, B) is stabilizable and (C, A) is observable, then it is straightforward to show that (A_1, B_1) is controllable, and both (C_1, A_1) and (C_2, A_2) are observable.

We let G_1, G_2, X_{z_k}, and z_k be defined as in Lemma 8. We define $D_1 := \frac{1}{2}X_{-1} + \sum_{k=1}^{N} \frac{z_k}{1+z_k} X_{z_k}$. It then follows that $\lim_{z\to\infty}(G_1(z)) = D_1$ so D_1 is real. The transfer function may be written as $G = D + C_1(zI - A_1)^{-1}B_1 + C_2(zI - A_2)^{-1}B_2 = G_1 + G_2$, with the second equality coming from the definition of G_2. Recall that $\mathrm{spec}(A_1) \in \bar{\mathcal{C}}_+$, and $\mathrm{spec}(A_2) \in \mathcal{C}_-$ from the transformation defined above, and from Lemma 8 we have that the poles of G_2 are all in $\mathcal{C}_-$. It then follows that $G_1(z) = D_1 + C_1(zI - A_1)^{-1}B_1$ and $G_2(z) = D_2 + C_2(zI - A_2)^{-1}B_2$.

$G_1(z)$ was shown to be DPR in Lemma 8, and (A_1, B_1) is controllable, and (C_1, A_1) is observable. Hence, from the classical discrete-time positive real lemma (Lemma 2), there exists $P_1 > 0$, $L_1 \in \mathbb{R}^{\hat{r}\times\hat{d}}$ and $W_1 \in \mathbb{R}^{\hat{r}\times n}$ such that

$$\begin{bmatrix} P_1 - A_1^T P_1 A_1 & C_1^T - A_1^T P_1 B_1 \\ C_1 - B_1^T P_1 A_1 & D_1 + D_1^T - B_1^T P_1 B_1 \end{bmatrix} = \begin{bmatrix} L_1^T \\ W_1^T \end{bmatrix} [L_1 \ W_1] \geq 0. \quad (8)$$

Since $\mathrm{spec}(A_1) \in \bar{\mathcal{C}}_+$, $P_1 > 0$ and $P_1 - A_1^T P_1 A_1 \geq 0$, it can be shown that $\mathrm{spec}(A_1)$ is on the unit circle and A_1 is semisimple. [2]

To show condition 1 we note that $D_2 = D - D_1$ from the definition of D_1 above, and we have shown that there is a $P_1 > 0$ such that (8) holds. Now, if there is also a P_2 such that equation (7) in condition 1 holds then

$$\begin{bmatrix} P_1 - A_1^T P_1 A_1 & 0 & C_1^T - A_1^T P_1 B_1 \\ 0 & 0 & 0 \\ C_1 - B_1^T P_1 A_1 & 0 & D_1 + D_1^T - B_1^T P_1 B_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & P_2 - A_2^T P_2 A_2 & C_2^T - A_2^T P_2 B_2 \\ 0 & C_2 - B_2^T P_2 A_2 & D_2 + D_2^T - B_2^T P_2 B_2 \end{bmatrix} \geq 0, \quad (9)$$

and it can be verified that $P = T^T \mathrm{diag}\,(P_1 \ P_2)\, T > 0$ satisfies (4). This is most easily seen by pre- and post-multiplying (9) by $\begin{pmatrix} T^T & I \end{pmatrix}$ and its transpose respectively, and using the transformations defined in (6).

To see condition 2, we first partition $\hat{P} = (T^{-1})^T P T^{-1}$ compatibly with $\mathrm{diag}\,(A_1 \ A_2)$, and by pre-multiplying the top left hand block in (4) by $(T^{-1})^T$ and post-multiplying by T^{-1} we conclude that

$$\begin{bmatrix} \hat{P}_{11} - A_1^T \hat{P}_{11} A_1 & \hat{P}_{12} - A_1^T \hat{P}_{12} A_2 \\ \hat{P}_{12}^T - A_2^T \hat{P}_{12}^T A_1 & \hat{P}_{22} - A_2^T \hat{P}_{22} A_2 \end{bmatrix} \geq 0. \quad (10)$$

Here, $\hat{P}_{11} > 0$ and $\hat{P}_{11} - A_1^T \hat{P}_{11} A_1 \geq 0$. In the same way as before, we may conclude that A_1 is semisimple and $\mathrm{spec}(A_1)$ is on the unit circle. We next show that $\hat{P}_{11} - A_1^T \hat{P}_{11} A_1 = 0$. Denote the number of columns of A_1 by $\hat{d}$. Since the matrix A_1 is semisimple, then there exists a basis $\mathbf{v}_1, \ldots, \mathbf{v}_{\hat{d}}$ of $\mathbb{C}^{\hat{d}}$ and $z_1, \ldots, z_{\hat{d}} \in \mathbb{C}$ with $|z_k| = 1$ such that $A_1\mathbf{v}_k = z_k\mathbf{v}_k$ $(k = 1, \ldots, \hat{d})$. Accordingly, $\mathbf{v}_k^*(\hat{P}_{11} - A_1^T \hat{P}_{11} A_1)\mathbf{v}_k = (1 - |z_k|^2)\mathbf{v}_k^* \hat{P}_{11}\mathbf{v}_k = 0$, and since $\hat{P}_{11} - A_1^T \hat{P}_{11} A_1 \geq 0$ we have that $(\hat{P}_{11} - A_1^T \hat{P}_{11} A_1)\mathbf{v}_k = 0$. This must hold for each of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{\hat{d}}$, from which we conclude that $\hat{P}_{11} - A_1^T \hat{P}_{11} A_1 = 0$. It immediately follows from the non-negative definiteness of (10) that $\hat{P}_{12} - A_1^T \hat{P}_{12} A_2 = 0$. This is a special case of a Stein equation, which has the unique solution $\hat{P}_{12} = 0$ in this case (Jiang and Wei, 2003). We may therefore tidy up the notation and abbreviate $\hat{P}_{11}$ and $\hat{P}_{22}$ to P_1 and P_2 respectively, such that $\hat{P} = \mathrm{diag}\,(P_1 \ P_2)$.

We return to (4) and pre-multiply by $\mathrm{diag}\,\big((T^{-1})^T \ I\big)$ and post-multiply by $\mathrm{diag}\,\big(T^{-1} \ I\big)$ to find that

$$\begin{bmatrix} P_1 - A_1^T P_1 A_1 & 0 & C_1^T - A_1^T P_1 B_1 \\ 0 & P_2 - A_2^T P_2 A_2 & C_2^T - A_2^T P_2 B_2 \\ C_1 - B_1^T P_1 A_1 & C_2 - B_2^T P_2 A_2 & D + D^T - B_1^T P_1 B_1 - B_2^T P_2 B_2 \end{bmatrix} \geq 0,$$

where the non-negative definiteness and $P_1 - A_1^T P_1 A_1 = 0$ imply that $C_1^T - A_1^T P_1 B_1 = 0$ and P_2 satisfies (7) if $D_2 = D - \frac{1}{2}B_1^T P_1 B_1$.

Finally, it can be shown that P_1 is uniquely determined by these two equations. To see this, note that if $\tilde{P}_1$ also satisfies $\tilde{P}_1 - A_1^T \tilde{P}_1 A_1 = 0$ and $C_1^T - A_1^T \tilde{P}_1 B_1 = 0$, then $X := P_1 - \tilde{P}_1$ satisfies $X - A_1^T X A_1 = 0$ and $A_1^T X B_1 = 0$. Since $\mathrm{spec}(A_1)$ is on the unit circle, then A_1 is invertible, whereupon $XB_1 = 0$. But this implies that $XB_1 = A_1^T X A_1 B_1 = 0$, whereupon $XA_1B_1 = 0$. Proceeding by induction, we find that $XA_1^k B_1 = 0$ $(k = 0, 1, 2, \ldots)$. Since (A_1, B_1) is controllable, we conclude that $X = 0$ and so P_1 is uniquely determined.

Proof. Proof of Theorem 7. We prove the chain of implications $3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 3$.

$3 \Rightarrow 2$ This follows immediately from 3 (i–iv) and noting that

$$\begin{bmatrix} P_- - A^T P_- A & C^T - A^T P_- B \\ C - B^T P_- A & D + D^T - B^T P_- B \end{bmatrix} = \begin{bmatrix} L^T \\ W^T \end{bmatrix} [L \ W],$$

as required by 2.

$2 \Rightarrow 1$ That G is DPR is well known. This follows from the observation that, if $|z| > 1$, then by pre-multiplying (4) by $\begin{bmatrix} B^T(z^*I - A^T)^{-1} & I \end{bmatrix}$ and post-multiplying by $\mathrm{col}\,\big((zI - A)^{-1}B \ I\big)$, then we find by factorisation of the matrix equation that

$$G(z) + G^{\sim}(z) + B^T(z^*I - A^T)^{-1}P(zI - A)^{-1}B(1 - |z|^2) \geq 0,$$

which implies that $G(z) + G^{\sim}(z) \geq B^T(z^*I - A^T)^{-1}P(zI - A)^{-1}B(|z|^2 - 1) \geq 0$.

[2] Consider the system $\mathbf{x}(k+1) = A_1\mathbf{x}(k)$ and the non-increasing Lyapunov function

$$V(\mathbf{x}(k+1)) = \mathbf{x}(k+1)^T P_1 \mathbf{x}(k+1) = \mathbf{x}(k)^T A_1^T P_1 A_1 \mathbf{x}(k) \leq V(\mathbf{x}(k)),$$

where $V(\mathbf{x}(0)) \geq V(\mathbf{x}(N)) \geq 0$ for $N \in \mathbb{Z}_+$. But if A_1 has an eigenvalue in $\mathcal{C}_+$ then there will exist an $\mathbf{x}(0)$ such that $V(\mathbf{x}(N)) \to \infty$ as $N \to \infty$, hence the eigenvalues of A_1 are on the unit circle. It can similarly be shown that A_1 is semisimple (i.e., every Jordan block is of size 1).

Now to show that 1 part b holds we must demonstrate stabilizability of (A, B). This proceeds in a manner analogous to the proof of Lemma 9. By pre-multiplying the top left hand block in (4) by $(T^{-1})^T$ and post-multiplying by T^{-1}, and recalling (6), it follows that there exists $\hat{P}_{11} > 0, \hat{P}_{22} > 0$ and a real matrix $\hat{P}_{12}$ such that (10) holds, and it is easily shown that (C_1, A_1) is observable from the observability of (C, A). We also find that $\hat{P}_{11} - A_1^T \hat{P}_{11} A_1 = 0$, $\hat{P}_{12} = 0$, and $C_1^T - A_1^T \hat{P}_{11} B_1 = 0$.

Now, let $\lambda \in \mathbb{C}$ and $\mathbf{z} \in \mathbb{C}^d$ satisfy $\mathbf{z}^T B_1 = 0$ and $\mathbf{z}^T A_1 = \lambda \mathbf{z}^T$ (so $|\lambda| = 1$ since, following the proof of Lemma 9, we have that $\mathrm{spec}(A_1)$ is on the unit circle). We will show that $\mathbf{z} = 0$, which implies that (A_1, B_1) is controllable, and it is then easily verified that (A, B) is stabilizable since $\mathrm{spec}(A_2) \in \mathcal{C}_-$. Accordingly, note that, since $\hat{P}_{11} > 0$ and $\mathrm{spec}(A_1)$ is on the unit circle, then $\hat{P}_{11}$ and A_1 are nonsingular, and so $(A_1^T \hat{P}_{11})^{-1} = A_1 \hat{P}_{11}^{-1}$. Thus, $\mathbf{z}^T (A_1^T \hat{P}_{11})^{-1} = \mathbf{z}^T A_1 \hat{P}_{11}^{-1} = \lambda \mathbf{z}^T \hat{P}_{11}^{-1} = \lambda \mathbf{z}^T (A_1^T \hat{P}_{11})^{-1} A_1^T$, and so $\frac{1}{\lambda} \mathbf{z}^T (A_1^T \hat{P}_{11})^{-1} = \mathbf{z}^T (A_1^T \hat{P}_{11})^{-1} A_1^T$. Since, in addition, $\mathbf{z}^T (A_1^T \hat{P}_{11})^{-1} (C_1 - A_1^T \hat{P}_{11} B_1) = \mathbf{z}^T (A_1^T \hat{P}_{11})^{-1} C_1 = 0$, and (C_1, A_1) is observable, we conclude that $\mathbf{z}^T (A_1^T \hat{P}_{11})^{-1} = 0$, whence $\mathbf{z} = 0$, therefore (A_1, B_1) is controllable and (A, B) is stabilizable since $\mathrm{spec}(A_2) \in \mathcal{C}_-$.

We must now show that condition 1c holds given condition 2. We note that

$$\begin{aligned}
&[V\ U] \begin{bmatrix} P - A^T P A & C^T - A^T P B \\ C - B^T P A & D + D^T - B^T P B \end{bmatrix} \begin{bmatrix} V^\sim \\ U^\sim \end{bmatrix} \\
&= U(VC^T + UD^T)^\sim + (VC^T + UD^T)U^\sim.
\end{aligned}$$

Then, note that if $\mathbf{b} \in \mathbb{R}^n[z]$ satisfies $\mathbf{b}^T(U(VC^T + UD^T)^\sim + (VC^T + UD^T)U^\sim) = 0$, then for any given $\omega \in \mathbb{R}$ we have that

$$\mathbf{b}(e^{j\omega})^T \begin{bmatrix} V(e^{j\omega}) & U(e^{j\omega}) \end{bmatrix} \begin{bmatrix} P - A^T P A & C^T - A^T P B \\ C - B^T P A & D + D^T - B^T P B \end{bmatrix} \begin{bmatrix} V(e^{j\omega})^* \\ U(e^{j\omega})^* \end{bmatrix} = 0,$$

which, by (4), implies that $\mathbf{b}(e^{j\omega})^T (V(e^{j\omega})(P - A^T P A) + U(e^{j\omega})(C - B^T P A)) = 0$. Since this holds for all $\omega \in \mathbb{R}$, it follows that $\mathbf{b}^T(V(P - A^T P A) + U(C - B^T P A)) = 0$. Recalling that $U(z)B^T = V(z)(\frac{1}{z}I - A^T)$, and noting that $P - A^T P A = (\frac{1}{z}I - A^T)P(zI - A) + (\frac{1}{z}I - A^T)PA + A^T P(zI - A)$, we find that $\mathbf{b}^T(U(z)C + (V(z)A^T + U(z)B^T)P(zI - A)) = 0$.

1 ⇒ 3 First, let $T, A_1, A_2, B_1, B_2, C_1, C_2, D_1$ and D_2 be as in Lemma 9, and recall from the proofs of Lemmas 8 and 9 that there exists $P_1 > 0$ uniquely determined by the equations $P_1 - A_1^T P_1 A_1 = 0$ and $C_1^T - A_1^T P_1 B_1 = 0$ such that (8) holds, and that $G_2(z) := D_2 + C_2(zI - A_2)^{-1} B_2$ satisfies $G_2 + G_2^\sim = G + G^\sim$. From Lemma 9 it suffices to show that there exists $P_2 > 0$ satisfying (7) for there to exist the required $P > 0$.

From Lemma 4, because $G_2 + G_2^\sim = G + G^\sim \geq 0$ for all z such that $|z| = 1$ and z is not a pole of G, we have that there is a discrete-time spectral factor $Z \in \mathbb{R}^{r \times n}$ of $G + G^\sim$. Moreover, since $G_2 + G_2^\sim$ has no poles on the unit circle (Lemma 8), we have that Z has no poles on the unit circle. Define $K := UZ^\sim$, so $U(G + G^\sim)U^\sim = U(UD^T + VC^T)^\sim + (UD^T + VC^T)U^\sim = KK^\sim = UZ^\sim Z U^\sim$.

From the choice of Jordan normal form and the arguments made in previous lemmas, $\mathrm{spec}(A) \in \bar{\mathcal{C}}_-$. By definition, U, V are left coprime and $U(z)B^T = V(z)(\frac{1}{z}I - A^T)$, so $U(z)$ is nonsingular for all $z \in \mathcal{C}_-$. But $Z^\sim(z)$ has full column rank for all $z \in \mathcal{C}_-$, and neither U nor $Z^\sim$ have any poles in $\bar{\mathcal{C}}_-$, and we conclude that $K(z)$ has full column rank for all $z \in \mathcal{C}_-$ and K doesn't have any poles in $\bar{\mathcal{C}}_-$. This implies that $K^\sim(z)$ has full row rank for all $z \in \mathcal{C}_+$. Since, in addition, $KK^\sim = U(VC^T + DU^T)^\sim + (VC^T + DU^T)U^\sim$, which can only have poles at the origin and at infinity, then it is straightforward to verify that K must be polynomial.

Now, let $H = \mathrm{col}\,(H_1\ H_2) \in \mathbb{R}^{n \times n}[z]$ be a unimodular matrix where the rows of H_1 are a basis for the left syzygy of $U(VC^T + DU^T)^\sim + (VC^T + DU^T)U^\sim$. It follows from condition 1c that there exists a polynomial matrix X such that $(H_1 U)(z)C = X(z)(zI - A)$, and it is then easily shown that there exists a polynomial matrix $\hat{J}_1$ such that $(H_1 U)(z)C_2 = \hat{J}_1(z)(zI - A_2)$. Next, let E be a polynomial matrix and F be a real matrix such that $(H_2 U)(z)C_2 = E(z)(zI - A_2) + F$. The existence of such matrices follows from (Gantmacher, 1980, pp. 77–79). We then note that $H_2 K$ is polynomial and nonsingular for all $z \in \mathcal{C}_-$, and it follows from (Feinstein and Bar-Ness, 1980, Theorem II) that there exists a polynomial matrix $\hat{X}$ and a real matrix L_2 such that $(H_2 K)(z)L_2 + \hat{X}(z)(zI - A_2) = F$. We then let $\hat{J}_2 = E + \hat{X}$, and we find that $(H_2 K)(z)L_2 = (H_2 U)(z)C_2 - \hat{J}_2(zI - A_2)$. Finally, as $H_1 KK^\sim = H_1(U(VC^T + DU^T)^\sim + (VC^T + DU^T)U^\sim) = 0$, then $H_1 K = 0$.[3] Since, in addition, $(H_1 U)(z)C_2 - \hat{J}_1(z)(zI - A_2) = 0$, then, with the notation $J := H^{-1}\mathrm{col}\left(\hat{J}_1\ \hat{J}_2\right)$, we conclude that J is polynomial and $K(z)L_2 = U(z)C_2 - J(z)(zI - A_2)$.

We now let $P_2 := \sum_{k=0}^{\infty} (A_2^T)^k L_2^T L_2 A_2^k$, which, since $\mathrm{spec}(A_2) \in \mathcal{C}_-$, is the unique solution to the Lyapunov equation $P_2 - A_2^T P_2 A_2 = L_2^T L_2$, and satisfies $P_2 \geq 0$ (Zhou et al., 1996, Lemma 21.6). Moreover, we let $W_2 := \lim_{z \to \infty}(Z(z))$. We will show that $Z(z) - L_2(zI - A_2)^{-1} B_2$ has no poles in $\mathcal{C}_-$. Since $\mathrm{spec}(A_2) \in \mathcal{C}_-$, and the poles of Z are all in $\mathcal{C}_-$, then we conclude that $Z(z) = W_2 + L_2(zI - A_2)^{-1} B_2$.

To see that $Z(z) - L_2(zI - A_2)^{-1} B_2$ has no poles in $\mathcal{C}_-$, we recall that there exists a polynomial matrix J such that $K(z)L_2 = U(z)C_2 - J(z)(zI - A_2)$, and that $K(z) = UZ^\sim$ where $Z(z)$ is a spectral factor for $G(z) + G^\sim(z) = G_2(z) + G_2^\sim(z)$. It follows that $K(z)(Z(z) - L_2(zI - A_2)^{-1} B_2) = J(z)B_2 - U(z)C_2(zI - A_2)^{-1} B_2 + U(z)(G_2(z) + G_2^\sim(z)) = J(z)B_2 + U(z)(D_2 + D_2^T + B_2^T z(I - zA_2^T)^{-1} C_2^T)$, which has no poles in $\mathcal{C}_-$. Since, in addition, $K(z)$ has full column rank for all $z \in \mathcal{C}_-$, then it is easily shown that $Z(z) - L_2(zI - A_2)^{-1} B_2$ has no poles in $\mathcal{C}_-$.

Next, we note that $L_2^T L_2 = (\frac{1}{z} - A_2)^T P_2 A_2 + \frac{1}{z} P_2 (zI - A_2)$, and that there exists a polynomial matrix J such that $Z^\sim(z) L_2 (zI - A_2)^{-1} = C_2(zI - A_2)^{-1} - U(z)^{-1} J(z)$. We then find that $(W_2^T L_2 - C_2 + B_2^T P_2 A_2)(zI - A_2)^{-1} = (Z^\sim(z) L_2 - C_2 + B_2^T P_2 A_2)(zI - A_2)^{-1} - B_2^T(\frac{1}{z}I - A_2^T)^{-1} L_2^T L_2 (zI - A_2)^{-1} = -U(z)^{-1} J(z) - B_2^T(I -$

[3] To see this, note that $H_1 KK^\sim = 0$ implies that $H_1(e^{j\omega})K(e^{j\omega})(H_1(e^{j\omega})K(e^{j\omega}))^* = 0$ for all $\omega \in \mathbb{R}$, which implies that $H_1(e^{j\omega})K(e^{j\omega}) = 0$ for all $\omega \in \mathbb{R}$, whence $H_1 K = 0$.

$A_2^T z)^{-1}P_2$, which has no poles in $\mathcal{C}_-$. Since spec$(A_2) \in \mathcal{C}_-$, then it is easily shown that $W_2^T L_2 - C_2 + B_2^T P_2 A_2 = 0$.

Now, note that $(W_2^T + B_2^T(\frac{1}{z}I - A_2^T)^{-1}L_2^T)(W_2 + L_2(zI - A_2)^{-1}B_2) = G_2(z) + G_2^{\sim}(z)$, which implies that $W_2^T W_2 - (D_2 + D_2^T - B_2^T P_2 B_2) = B_2^T(P_2 - (\frac{1}{z}I - A_2^T)^{-1}L_2^T L_2(zI - A_2)^{-1})B_2 + (C_2 - W_2^2 L_2)(zI - A_2)^{-1}B_2 + B_2^T(\frac{1}{z}I - A_2^T)^{-1}(C_2^T - L_2^T W_2)$. Noting that $(\frac{1}{z}I - A_2^T)^{-1}L_2^T L_2(zI - A_2)^{-1} = P_2 A_2(zI - A_2)^{-1} + \frac{1}{z}(\frac{1}{z}I - A_2^T)^{-1}P_2$, and recalling that $C_2 - W_2^T L_2 = B_2^T P_2 A_2$, we find that $W_2^T W_2 - (D_2 + D_2^T - B_2^T P_2 B_2) = 0$.

We have constructed a $P_2 \geq 0$ such that $P_2 - A_2^T P_2 A_2 = L_2^T L_2$, $C_2 - B_2^T P_2 A_2 = W_2^T L_2$, and $D_2 + D_2^T - B_2^T P_2 B_2 = W_2^T W_2$, in addition to a $P_1 > 0$ satisfying $P_1 - A_1^T P_1 A_1 = 0$, $C_1 - B_1^T P_1 A_1 = 0$, and $D_1 + D_1^T - B_1^T P_1 B_1 = 0$. Direct calculation then verifies that $P := T^T \mathrm{diag}\,(P_1\ P_2)\, T \geq 0$, $L := [0\ L_2]\, T$ and $W := W_2$ satisfy conditions 3(ii)–(v) of the present theorem statement.

Finally, we must show that if P is as in condition 2, then $P > 0$ as opposed to being only non-negative definite (and it can similarly be shown that $P_- > 0$ whenever P_- is as in condition 3). Accordingly, we must show that $P\mathbf{z} = 0$ implies $\mathbf{z} = 0$. Consider that $P\mathbf{z} = 0$, so $\mathbf{z}^T(P - A^T PA)\mathbf{z} = -(A\mathbf{z})^T P(A\mathbf{z}) \geq 0$, hence $P(A\mathbf{z}) = 0$. Proceeding by induction we find that $PA^k\mathbf{z} = 0$ for $k = 0, 1, 2, \ldots$, which by the non-negative definiteness of (4) implies that $(C - B^T PA))(A^k\mathbf{z}) = 0$, and hence $CA^k\mathbf{z} = 0$ for $k = 0, 1, 2, \ldots$. But (C, A) is observable so it must be that $\mathbf{z} = 0$, demonstrating that $P > 0$.

5. PROOF OF THEOREM 5, AN ASSUMPTION-FREE THEOREM ON DPR SYSTEMS

We have now proved Theorem 7, the proof of which shall be drawn upon in the proof of Theorem 5, the main result of this paper. Owing to space constraints, only a sketch of the proof is provided here, with the full details to follow in a subsequent paper.

Theorem 7 can be proved from Theorem 5 by relating the system considered in Theorem 5 to the observable subsystem obtained from the staircase observability form:

Lemma 10. Let $\tilde{A} \in \mathbb{R}^{\tilde{d}\times\tilde{d}}$, $\tilde{B} \in \mathbb{R}^{\tilde{d}\times\tilde{n}}$, $\tilde{C} \in \mathbb{C}^{\tilde{n}\times\tilde{d}}$, and $\tilde{D} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}$. There exists a nonsingular $T \in \mathbb{R}^{\tilde{d}\times\tilde{d}}$ such that

$$T\tilde{A}T^{-1} = \underbrace{\begin{bmatrix} A & 0 \\ A_{21} & A_{22} \end{bmatrix}}_{\hat{A}}, T\tilde{B} = \underbrace{\begin{bmatrix} B \\ B_2 \end{bmatrix}}_{\hat{B}}, \tilde{C}T^{-1} = \underbrace{[C\ 0]}_{\hat{C}}, \tilde{D} = D,$$

where (C, A) is observable (Polderman and Willems, 1998). This is often called the staircase observability form.

We shall refer to $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ as the whole, unobservable, system, while A, B, C, and D shall be referred to as the observable part of the system. Next, we state the following lemma that shall be used to connect results in the observable case to the unobservable case: if condition 2 of Theorem 7 holds for the observable part of the system then condition 2 holds for the entire system, and vice versa.

Lemma 11. Let $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, A, B, C, D$ be as in Lemma 10. The following conditions hold

(1) If there exists a $\tilde{P} \geq 0$ such that

$$\begin{bmatrix} \tilde{P} - \tilde{A}^T\tilde{P}\tilde{A} & \tilde{C}^T - \tilde{A}^T\tilde{P}\tilde{B} \\ \tilde{C} - \tilde{B}^T\tilde{P}\tilde{A} & \tilde{D} + \tilde{D}^T - \tilde{B}^T\tilde{P}\tilde{B} \end{bmatrix} \geq 0, \quad (11)$$

then there exists $P \geq 0$ such that (4) holds.

(2) If there exists $P \geq 0$ such that (4) holds then there exists $\tilde{P} \geq 0$ such that (11) holds.

Proof. Let $T, \hat{A}, \hat{B}$ and $\hat{C}$ be as in Lemma 10, let $\hat{D} = D$, and let $\hat{P} = (T^{-1})^T\tilde{P}T^{-1}$, then

$$\begin{bmatrix} (T^{-1})^T & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \tilde{P} - \tilde{A}^T\tilde{P}\tilde{A} & \tilde{C}^T - \tilde{A}^T\tilde{P}\tilde{B} \\ \tilde{C} - \tilde{B}^T\tilde{P}\tilde{A} & \tilde{D} + \tilde{D}^T - \tilde{B}^T\tilde{P}\tilde{B} \end{bmatrix}\begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} \hat{P} - \hat{A}^T\hat{P}\hat{A} & \hat{C}^T - \hat{A}^T\hat{P}\hat{B} \\ \hat{C} - \hat{B}\hat{P}\hat{A} & \hat{D} + \hat{D}^T - \hat{B}^T\hat{P}\hat{B} \end{bmatrix} \geq 0. \quad (12)$$

Now partition $\hat{P}$ compatibly with $\hat{A}$ as $\begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{12}^T & \hat{P}_{22} \end{bmatrix}$, and let $\hat{P}_{22}^\dagger$ denote the pseudo-inverse of $\hat{P}_{22}$ (so $\hat{P}_{22}^\dagger\hat{P}_{22}\hat{P}_{22}^\dagger = \hat{P}_{22}^\dagger$, $\hat{P}_{22}\hat{P}_{22}^\dagger\hat{P}_{22} = \hat{P}_{22}$, and $(\hat{P}_{22}^\dagger)^T = \hat{P}_{22}^\dagger$ since $\hat{P}_{22}$ is symmetric). Then let $\hat{T} = \begin{bmatrix} I & 0 \\ \hat{P}_{22}^\dagger\hat{P}_{12}^T & I \end{bmatrix}$ such that

$$\hat{T}\hat{A}\hat{T}^{-1} = \underbrace{\begin{bmatrix} A & 0 \\ \check{A}_{21} & \check{A}_{22} \end{bmatrix}}_{\check{A}}, \hat{T}\hat{B} = \underbrace{\begin{bmatrix} B \\ \check{B}_2 \end{bmatrix}}_{\check{B}}, \hat{C}\hat{T}_{-1} = \underbrace{[C\ 0]}_{\check{C}}, \hat{D} = \check{D}.$$

It can be shown that $(I - \hat{P}_{22}\hat{P}_{22}^\dagger)\hat{P}_{12}^T = 0$. This is because $\hat{P} \geq 0$, so $\mathbf{z}^T\hat{P}_{22} = 0$ implies $\mathbf{z}^T\hat{P}_{12} = 0$, meaning the left null space of $\hat{P}_{12}$ is a subset of the left null space of $\hat{P}_{22}$. In turn, this means that the range space of $\hat{P}_{12}^T$ is a subset of the range space of $\hat{P}_{22}$, so there exists a matrix X such that $\hat{P}_{12}^T = \hat{P}_{22}X$. This implies that $(I - \hat{P}_{22}\hat{P}_{22}^\dagger)\hat{P}_{12}^T = (I - \hat{P}_{22}\hat{P}_{22}^\dagger)\hat{P}_{22}X = 0$. It then follows that

$$(\hat{T}^{-1})^T\hat{P}\hat{T}^{-1} = \begin{bmatrix} \hat{P}_{11} - \hat{P}_{12}\hat{P}_{22}^\dagger\hat{P}_{12}^T & 0 \\ 0 & \hat{P}_{22} \end{bmatrix}. \quad (13)$$

After introducing the notation $P = \hat{P}_{11} - \hat{P}_{12}\hat{P}_{22}^\dagger\hat{P}_{12}^T$, we may write

$$\begin{bmatrix} (\hat{T}^{-1})^T & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \hat{P} - \hat{A}^T\hat{P}\hat{A} & \hat{C}^T - \hat{A}^T\hat{P}\hat{B} \\ \hat{C} - \hat{B}\hat{P}\hat{A} & \hat{D} + \hat{D}^T - \hat{B}^T\hat{P}\hat{B} \end{bmatrix}\begin{bmatrix} \hat{T}^{-1} & 0 \\ 0 & I \end{bmatrix} \geq 0,$$

from which we have

$$\begin{bmatrix} P - A^T PA & C^T - A^T PB \\ C - B^T PA & D + D^T - B^T PB \end{bmatrix} \geq \begin{bmatrix} \check{A}_{21}^T \\ \check{B}_2^T \end{bmatrix}\hat{P}_{22}\begin{bmatrix} \check{A}_{21} & \check{B}_2 \end{bmatrix} \geq 0,$$

since $\hat{P}_{22} \geq 0$, and from (13) we have that $P = \hat{P}_{11} - \hat{P}_{12}\hat{P}_{22}^\dagger\hat{P}_{12}^T \geq 0$.

As for condition 2, if $P \geq 0$ satisfies (4) then it can be verified that $\tilde{P} = T^T\mathrm{diag}(P, 0)T$ satisfies (11).

Armed with Lemma 11 and the proof of Theorem 7 we can finally sketch the proof of Theorem 5, the full details of which will be provided in a subsequent paper.

Proof. Proof of Theorem 5. The proof proceeds by proving the chain of implications $3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 3$.

$\mathbf{3} \Rightarrow \mathbf{2}$ This follows immediately, as it did in the proof of Theorem 7.

Proving the remaining implications proceeds by relating each of the conditions of Theorem 5 concerning the whole unobservable system to the corresponding conditions of Theorem 7 concerning the observable part. The full details will follow in a subsequent paper.

6. CONCLUSION

We have proved a theorem on the discrete-time positive real lemma that does not require controllability and observability. This result will prove useful in further work. In particular, the discrete-time positive real lemma is closely linked to system passivity and the related optimal control problem, and similar uncontrollable and unobservable results are obtainable here, though this was beyond the scope of this paper. Also alluded to earlier, achieving similar results that remove assumptions of controllability and observability is also possible in the discrete-time bounded real lemma, as will be presented in a sequel paper.

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